

# On the Komar Energy and the Generalized Smarr Formula for a Charged Black Hole of Noncommutative Geometry

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We calculate the Komar energy  $E$  for a charged black hole inspired by noncommutative geometry and identify the total mass ( $M_0$ ) by considering the asymptotic limit. We also found the generalized Smarr formula, which shows a deformation from the well known relation  $M_0 - \frac{Q_0^2}{r} = 2ST$  depending on the noncommutative scale length  $\ell$ .

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## I. INTRODUCTION

There is a deep connection between gravity and thermodynamics that has been known for a long time, from the works of Bekenstein and Hawking [1–3] to the recent research of Padmanabhan [4, 5]. In a thermodynamical system like Schwarzschild black hole, the entropy  $S$ , Hawking temperature  $T$  and energy  $E$  are related by the first law of thermodynamics

$$dE = TdS, \quad (1)$$

where  $E$  is identified with the Komar energy [6, 7] and specifically for a Schwarzschild black hole it equals the total mass of the black hole,  $M$ . There is also an integral version of this equation

$$E = M = 2TS. \quad (2)$$

known as the Smarr formula [8] and it can be verified by putting the expressions for entropy and the temperature

$$T = \frac{1}{8\pi M} \quad (3)$$

$$S = \frac{A}{4} = 4\pi M^2. \quad (4)$$

Eq. (2) has been obtained in different ways [5, 9] and the Komar energy is identified with the conserved charge associated with the Killing vector defined at the event horizon (see for example [10]). Recently, some generalised expressions for Smarr formula in different spacetimes have been studied [9–11] and in particular, the Kerr-Newman black hole with electric charge  $Q$  and angular momentum  $J$  satisfies the Smarr relation[12]

$$M = 2TS + \Phi_H Q + 2\Omega_H J \quad (5)$$

where  $\Phi_H$  and  $\Omega_H$  are the electric potential and angular velocity at the horizon, respectively.

In this paper we investigate the specific case of a spherically symmetric charged black hole inspired by noncommutative geometry [13–20]. This solution is obtained by introducing the noncommutativity effect through a coherent state formalism [21–23], which implies the replacement of the point distributions by smeared structures throughout a region of linear size  $\ell$ . We perform the analysis by obtaining the Komar energy by direct integration and found the generalized Smarr formula, which shows a deformation from the usual relation depending on the noncommutative parameter  $\ell$ .

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## II. KOMAR ENERGY OF THE CHARGED NONCOMMUTATIVE BLACK HOLE

Many formulations of noncommutative field theory are based on the Weyl-Wigner-Moyal  $*$ -product [24–26] that lead to some important problems such as Lorentz invariance breaking, loss of unitarity or UV divergences of the quantum field theory. However, Smailagic and Spallucci [13–17, 19] explained recently a model of noncommutativity that can be free from the problems mentioned above. They assume that a point-like mass  $M$  and charge  $Q$ , instead of being quite localized at a point, must be described by a smeared structure throughout a region of linear size  $\ell$ . The metric for this distribution is given by, [20],

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \quad (6)$$

where

$$f(r) = 1 - \frac{2M(r)}{r} + \frac{Q^2(r)}{r^2} \quad (7)$$

$$Q(r) = \frac{Q_0}{\sqrt{\pi}} \sqrt{\gamma^2 \left( \frac{1}{2}, \frac{r^2}{4\ell^2} \right) - \frac{r}{\sqrt{2}\ell} \gamma \left( \frac{1}{2}, \frac{r^2}{2\ell^2} \right) + \frac{\sqrt{2}r}{\ell} \gamma \left( \frac{3}{2}, \frac{r^2}{4\ell^2} \right)} \quad (8)$$

$$M(r) = \frac{2M_0}{\sqrt{\pi}} \gamma \left( \frac{3}{2}, \frac{r^2}{4\ell^2} \right) \quad (9)$$

and

$$\gamma \left( \frac{a}{b}, x \right) = \int_0^x du u^{\frac{a}{b}-1} e^{-u} \quad (10)$$

is the lower incomplete gamma function. Considering a spatial 2-sphere  $V$  with boundary  $\partial V$ , the Komar integral for the energy is

$$E(V) = \frac{a}{16\pi} \oint_{\partial V} \nabla^\mu \xi^\nu d\Sigma_{\mu\nu} \quad (11)$$

where the killing vector is  $\xi = \frac{\partial}{\partial t}$ ,  $d\Sigma_{\mu\nu}$  is the surface element at the boundary and the value of constant  $a$  will be found by comparison with the noncommutative Schwarzschild case. This is

$$E = \frac{2a}{16\pi} \oint_{\partial V} \nabla^\mu \xi^t d\Sigma_{\mu t}, \quad (12)$$

where the factor 2 appears because of the symmetry of the integrand. The covariant derivative involved is

$$\nabla_\mu \xi^t = \partial_\mu \xi^t + \Gamma_{\mu\sigma}^t \xi^\sigma = \Gamma_{\mu t}^t, \quad (13)$$

and for the noncommutative charged solution the nonvanishing connections are

$$\Gamma_{rt}^t = \frac{-\frac{dM}{dr} r^2 + rM + \frac{r}{2} \frac{dQ^2}{dr} - Q^2}{r(r^2 - 2Mr + Q^2)} \quad (14)$$

$$\Gamma_{tt}^t = \Gamma_{\theta t}^t = \Gamma_{\varphi t}^t = 0, \quad (15)$$

giving

$$E = \frac{a}{8\pi} \oint_{\partial V} \frac{-\frac{dM}{dr} r^2 + rM + \frac{r}{2} \frac{dQ^2}{dr} - Q^2}{r^3} d\Sigma_{rt}. \quad (16)$$

The surface element corresponds to

$$d\Sigma_{rt} = -d\Sigma_{tr} = -r^2 \sin^2 \theta d\theta d\varphi. \quad (17)$$

and therefore

$$E = -\frac{a}{8\pi} \frac{-\frac{dM}{dr} r^2 + rM + \frac{r}{2} \frac{dQ^2}{dr} - Q^2}{r} \oint_{\partial V} \sin^2 \theta d\theta d\varphi \quad (18)$$

$$E = \frac{a}{2} \left[ \frac{dM}{dr} r - M - \frac{1}{2} \frac{dQ^2}{dr} + \frac{Q^2}{r} \right]. \quad (19)$$

By comparison with the Komar energy of the Schwarzschild black hole, we shall identify  $a = -2$ . Hence, the energy of the noncommutative charged black hole is finally given by

$$E = M - \frac{dM}{dr} r - \frac{Q^2}{r} + Q \frac{dQ}{dr}. \quad (20)$$

The horizons of the metric (6) can be found by setting  $f(r_{\pm}) = 0$ , i.e.

$$r_{\pm}^2 - 2r_{\pm}M(r_{\pm}) + Q^2(r_{\pm}) = 0, \quad (21)$$

which can be written as

$$r_{\pm} = M(r_{\pm}) \pm \sqrt{M^2(r_{\pm}) - Q^2(r_{\pm})}. \quad (22)$$

Hawking temperature is defined in terms of the surface gravity at the event horizon by

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \partial_r f(r)|_{r=r_+}, \quad (23)$$

which gives in this case

$$T = \frac{1}{2\pi r_+^2} \left[ M(r_+) - \frac{Q^2(r_+)}{r_+} - r_+ \frac{dM}{dr} \Big|_{r=r_+} + Q(r_+) \frac{dQ}{dr} \Big|_{r=r_+} \right]. \quad (24)$$

The entropy in terms of the area of the horizon is given by the well known relation

$$S = \frac{A}{4} = \pi r_+^2 \quad (25)$$

and therefore, the Komar energy (20) at the event horizon becomes

$$E = 2\pi r_+^2 T = 2ST. \quad (26)$$

Using the value  $r_{\pm} = M_0 \pm \sqrt{M_0^2 - Q_0^2}$  as a first approximation of the horizons (22) and putting them into the incomplete gamma functions of relations (8) and (9) one obtains

$$r_{\pm} = M_{\pm} \pm \sqrt{M_{\pm}^2 - Q_{\pm}^2} \quad (27)$$

where we have defined

$$M_{\pm} = M_0 \left[ \varepsilon \left( \frac{M_0 \pm \sqrt{M_0^2 - Q_0^2}}{2\ell} \right) - \frac{M_0 \pm \sqrt{M_0^2 - Q_0^2}}{\sqrt{\pi}\ell} \exp \left( -\frac{\left( M_0 \pm \sqrt{M_0^2 - Q_0^2} \right)^2}{4\ell^2} \right) \right] \quad (28)$$

$$Q_{\pm} = Q_0 \sqrt{\varepsilon^2 \left( \frac{M_0 \pm \sqrt{M_0^2 - Q_0^2}}{2\ell} \right) - \frac{\left( M_0 \pm \sqrt{M_0^2 - Q_0^2} \right)^2}{\sqrt{2\pi}\ell^2} \exp \left( -\frac{\left( M_0 \pm \sqrt{M_0^2 - Q_0^2} \right)^2}{4\ell^2} \right)} \quad (29)$$

and  $\varepsilon(x)$  is the Gauss error function,

$$\varepsilon(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du. \quad (30)$$

For a large value of its argument (i.e. large masses), function  $\varepsilon$  tends to unity while the exponential term goes to zero, giving the classical Reissner-Nordström horizons  $r_{\pm} \rightarrow r_{RN\pm} = M_0 \pm \sqrt{M_0^2 - Q_0^2}$ .

Using the same value as a first approximation for the event horizon in the Hawking temperature (23) one obtains [28]

$$T \approx \frac{1}{4\pi} \frac{r_+ - r_-}{r_+^2}. \quad (31)$$

This approximation permit us to write the Komar energy at the horizon, using Eqs.(26), (31) and (27), as

$$E = 2\pi r_+^2 T = \frac{r_+ - r_-}{2} \quad (32)$$

$$E = \frac{1}{2} \left[ M_+ + M_- + \sqrt{M_+^2 - Q_+^2} - \sqrt{M_-^2 - Q_-^2} \right]. \quad (33)$$

By considering the behavior of the functions  $M_{\pm}$  and  $Q_{\pm}$ , it is easy to see that the limit of large masses of (33), as well as taking the limit  $\ell \rightarrow 0$ , recover the Reissner-Nordström energy and for  $Q_0 = 0$  it gives the result of Banerjee and Gangopadhyay [27] for the noncommutative Schwarzschild black hole with the usual  $E = M_0$  that let us identify the quantity  $M_0$  as the total mass of the black hole and  $Q_0$  as its total electric charge.

With a similar procedure, the entropy can be approximated by

$$S = \pi r_+^2 \approx \pi \left( M_+ + \sqrt{M_+^2 - Q_+^2} \right)^2, \quad (34)$$

which give in the limit of large masses, or in the limit  $\ell \rightarrow 0$ , the usual result for the Reissner-Nordström black hole,  $S \rightarrow S_{RN} = \pi \left( M_0 + \sqrt{M_0^2 - Q_0^2} \right)^2$ .

Using Eqs. (8) and (9) and the property of the gamma function

$$\frac{\partial}{\partial u} \gamma \left( \frac{a}{b}, u \right) = e^{-u} u^{-1+\frac{a}{b}} \quad (35)$$

to perform the derivatives, the Komar energy (20) for this spacetime yields

$$E = M(r) - \frac{Q^2(r)}{r} - \frac{M_0}{2\sqrt{\pi}} \frac{r^3}{\ell^3} e^{-\frac{r^2}{4\ell^2}} + \frac{Q_0^2}{2\pi} \left[ \frac{2}{\ell} e^{-\frac{r^2}{4\ell^2}} \gamma \left( \frac{1}{2}, \frac{r^2}{4\ell^2} \right) - \frac{1}{\sqrt{2}\ell} \gamma \left( \frac{1}{2}, \frac{r^2}{2\ell^2} \right) + \frac{\sqrt{2}}{\ell} \gamma \left( \frac{3}{2}, \frac{r^2}{4\ell^2} \right) - \frac{r}{\ell^2} e^{-\frac{r^2}{2\ell^2}} + \frac{\sqrt{2}}{4} \frac{r^3}{\ell^4} e^{-\frac{r^2}{4\ell^2}} \right]. \quad (36)$$

Using the long distance approximations for the gamma functions

$$\gamma \left( \frac{3}{2}, \frac{r^2}{4\ell^2} \right) \simeq \frac{\sqrt{\pi}}{2} - \frac{r}{2\ell} e^{-r^2/4\ell^2} \quad (37)$$

$$\gamma\left(\frac{1}{2}, \frac{r^2}{2\ell^2}\right) \simeq \sqrt{\pi} - \sqrt{2}\ell \frac{e^{-r^2/2\ell^2}}{r} \quad (38)$$

$$\gamma\left(\frac{1}{2}, \frac{r^2}{4\ell^2}\right) \simeq \sqrt{\pi} - 2\ell \frac{e^{-r^2/4\ell^2}}{r} \quad (39)$$

we obtain the relation

$$\begin{aligned} M_0 - \frac{Q_0^2}{r} = & 2TS + \frac{M_0}{\sqrt{\pi}} \frac{r}{\ell} e^{-\frac{r^2}{4\ell^2}} \left(1 + \frac{r^2}{2\ell^2}\right) \\ & + \frac{Q_0^2}{\pi r} \left[ e^{-\frac{r^2}{2\ell^2}} \left(\frac{5}{2} + \frac{r^2}{2\ell^2} + \frac{4\ell^2}{r^2}\right) - e^{-\frac{r^2}{4\ell^2}} \left(4\sqrt{\pi}\frac{\ell}{r} + \sqrt{\pi}\frac{r}{\ell} + \frac{\sqrt{2}}{4}\frac{r^2}{\ell^2} + \frac{\sqrt{2}}{8}\frac{r^4}{\ell^4}\right) \right]. \end{aligned} \quad (40)$$

Since  $M_0$  and  $Q_0$  have been identified as the mass and charge of the black hole, Eq.(40) corresponds to the generalization of the Smarr formula for the noncommutative charged black hole. Note that this relation deviates from the usual one (5) by the two last terms in the right hand side, but it is clear that in the limit  $\ell \rightarrow 0$  these terms disappear. In the case  $Q_0 = 0$  we recover the relation for the noncommutative Schwarzschild black hole presented in [27, 29, 30].

### III. CONCLUSION

We have computed the Komar energy for a charged black hole inspired in noncommutative geometry and its asymptotic limit that let us identify the constant  $M_0$  as its total mass and  $Q_0$  as its electric charge. With these results, we obtain the noncommutative version of the Smarr formula (40) which show a deformation from the usual relation and the new terms depend on the noncommutative parameter  $\ell$ .

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